Some Types of Integral Extensions

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In this note we discuss some aspects of integral extensions and normality of related algebraic objects. Here we recall some results on ideal transforms. Let I be an ideal of a Noetherian ring R and N an R-module. Then the I-transform of N is defined by $T_I(N) = \varinjlim Hom_R(I^n, N)$. Let R be a local domain with maximal ideal M, and let R^* be its completion. The maximal ideal transformation $T(R) = T_M(R)$ of R is described as the set of elements x in K, the quotient field of R, with $M^n x \in R$ for some n. Then $NT(R) = T(R)/R$ is canonically regarded as a module over R^* and is isomorphic to $NT(R) \otimes R^* = NT(R^*)$.

We are interested in submodules of $NT(R)$ which correspond to certain ring extensions of R. As our starting point, we examine the Krull Akizuki Theorem(cf. [5], (33. 2)) and some idea of its proof. The following is a slight variation of [6, (3. 11)] .

(1) *Let R be a Noetherian ring with total quotient ring K and let S be a ring extension in K such that A/R is an Artinian module over R for any finite R-submodule A of S. Then every regular ideal of S is finitely generated.*

In fact, taking S for N(resp. R for M), the conditions required in [6], (3. 11) are all satisfied from our assumption. For a regular ideal I, let a be any regular element of I. Then S/Sa , hence I/Sa is a finite module over R, and I is a finitely generated ideal of S.

The Krull-Akizuki Theorem proved in [5] is stated as follows :

(*) Let R be a Noetherian integral domain with quotient field K, let L be a finite algebraic extension of K and let R' be a ring such that $R \subseteq R' \subseteq L$. If altitude $R = 1$, then for any ideal I' of R' such that $I' \neq 0$, R'/I' is a module of finite length over $R/I' \cap R$. In particular, R' is a Noetherian ring of altitude at most one.

To apply (1) to this theorem, we need to construct a finite extension R_1 in R' with the same quotient field as that of R'. Then taking R' for S(resp. R_1 for R), we get a proof of this theorem. The proof presented in [5] uses [5, (33. 1)], which is interesting by itself. Here we restate this in the following form :

(*) Let R be an Noetherian domain contained in an integral domain S with the quotient field finite algebraic over that of R. If the integral closure R' of R in S is Noetherian and $SpecS \to SpecR'$ is surjective, then S is integral over R, that is, $R' = S$.

For simplicity, we treat the case of integral domains. Let R be an integral domain with quotient field K contained in an integral domain S with quotient field L. Put $X = SpecR$ and $X' = SpecS$.

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(2) *The following conditions are equivalent.*

(a) S *is integral over* R.

(b) *For any integral domain* T *with* $R \subseteq T \subseteq L$, *the lying over theorem holds for the ring extension* $T \subseteq TS$.

Here we assume that R *is a Noetherian domain. Then these conditions are equivalent to*

(c) *For any* $P \in \bar{A}^*(R)$, *every element of* S *is integral over* R_P , *where* $\bar{A}^*(R)$ *denotes the set of asymptotic prime ideals of* R :

$\bar{A}^*(R) = \{P | PR_P^*$ *is minimal over* $aR + \bar{P}$ *where* $0 \neq a \in R$ *and* $\bar{P}$ *is a minimal prime in* $R_P^*\}$

In fact, the assertion (a) $\Rightarrow$ (b) is clear. Suppose (b) holds. Let V be a valuation ring of L with $R \subseteq V$. Then from our assumption the maximal ideal of VS is lying over the maximal ideal of V. Since the valuation ring is maximal with respect to the relation of domination, we have $VS = V$, hence $S \subseteq V$. As is well known, this implies that S is integral over R. The assertion (a) $\Rightarrow$ (c) is trivial. Conversely suppose (c) holds. Let x be any element of S, $T = K(x)$ and A the integral closure of R in T.

Then there is a finite integral extension B of R with quotient field T. It is easy to see that for any $P' \in \bar{A}^*(B)$, $P' \cap R$ is an element of $\bar{A}^*(R)$ and this defines the surjective map $\bar{A}^*(B) \to \bar{A}^*(R)$. For any $P' \in \bar{A}^*(B)$, $B_{P'}$ contains R_P with $P = P' \cap R$ and x is integral over $B_{P'}$. Since every height 1 prime ideal of Krull domain A lies over some $P' \in \bar{A}^*(B)$, our assumption implies that x is contained in A. Thus every element of S is integral over R.

Remark : If for any scheme Y over X, $X' \times_X Y \to Y$ is a closed map, then X' is integral over X. The condition (b) is a prototype of the above geometric form.

Let (R, M) be a local domain with dim $R \geq 1$.

(*) *There is one to one correspondence between the set of ring extensions of* R *in the maximal ideal transform* $T(R)$ *and that of* R^* *in* $T(R^*)$. *In fact,* $T(R) \otimes_R R^* \cong T(R^*)$ *and* $T(R)/R$ *is regarded as a module over* R^*, *canonically isomorphic to* $T(R^*)/R^*$. *If* S' *corresponds to* S, *then* S' *is isomorphic to* $S \otimes_R R^*$. *In particular, if* S *is finite integral over* R, *then* S' *is regarded as the completion of* S.

(3) *Suppose that the zero ideal of* R^*, *the completion of* R, *has the following irredundant decomposition by isolated components :* $(0) = I_0 \cap I_1 \cap I_2 \cap \cdots \cap I_n$, *where* I_i *is unmixed with height* $I_i = 0$ *and coheight* $I_i = 1 \, (1 \leq i \leq n)$.

Then we have a finite ring extension S *such that* S *has the maximal ideals* $M_0, M_1, \cdots, M_n$ *with height* $M_i = 1 \, (1 \leq i \leq n)$, *which correspond to* $I_0, I_1, \cdots I_n$ *respectively.*

In fact, let $I^{(i)}$ be the intersection of I_k , $k \neq i$. Then the sum of the ideals $I^{(0)}, I^{(1)}, \cdots I^{(n)}$ is M-primary ideal. In particular, we have a regular element $a = c_0 + c_1 + \cdots + c_n$ with $c_i \in I^{(i)}$ $(0 \leq i \leq n)$. Letting $e_i = c_i/a \, (0 \leq i \leq n)$, $1 = e_0 + e_1 + \cdots + e_n$, where the e_i are orthogonal idempotent elements. Corresponding to this decomposition, we have a ring extension $S' = R_0 \oplus R_1 \oplus \cdots \oplus R_n$,

$R_i = R^*e_i$ $0 \le i \le n$ in the total quotient ring of R^*. Then the ideal $J = R^*a + I^{(0)}$ is an MR^*- primary ideal with $JR_i \subseteq R^*$, $(1 \le i \le n)$ and hence $JR_0 \subseteq R^*$ since $e_0 = 1 - e_1 - \cdots - e_n$. Thus S' is a finite integral extension of R^* in $T(R^*)$ and it corresponds to a finite integral extension S in $T(R)$. Since $R_i \cong R^*/I_i$ and the completion S^* is isomorphic to the product $S' = R_0 \oplus R_1 \oplus \cdots \oplus R_n$, S has the maximal ideals M_0 , $\cdots$, M_n stated as above.

As a simple application we have the following result due to Ratliff (cf. [3], (3. 19)).

(4) *Suppose that the zero ideal of* R^* *has n coheight 1 minimal prime ideals. Then the integral closure of* R *in* K *has n height 1 maximal ideals.*

More precisely, we have the following :

(5) *Suppose that the zero ideal of* R^* *has a coheight 1 minimal prime ideal* P. *Then* R^*/P *is a 1-dimensional complete local domain, and the normalization* V^* *of* R^*/P *is a Noetherian valuation ring. Let* $V = V^* \cap K$ *where* K *is the quotient field of* R, *and let* R' *be the normalization of* R *in* K. *Then the contraction M of the maximal ideal of V to* R' *is the height* 1 *maximal prime ideal which corresponds to* P *as stated in* (3) *and* $V = R'_M$.

As for the integral closure of R in $T(R)$, we have a beautiful result, an extension of the Kull-Akizuki Theorem. Here we give a slight different proof to the following extension (cf. [6], (3. 15)).

(6) *Let* B *be the integral closure of* R *in* $T(R)$. *Then* $B = T(R) \cap V_1 \cap \cdots \cap V_n$, *where the* V_i *are Noetherian valuation rings and* n *is the number of coheight* 1 *minimal prime ideals of* R^*. *The integral closure in* K *is* $R' = R_1 \cap \cdots \cap R_r \cap V_1 \cap \cdots \cap V_n$, *where the* R_i *are quasi-local domains with dim* $R_i \ge 2$ *and* $r \le$ *the number of minimal prime ideals of coheight* ≥ 2 *in* R^*($r =$ *the the number of minimal prime ideals of coheight* ≥ 2 *in the Henselization of* R).

In fact, with the same notation as in (3), let $\sqrt{I_i} = P_i$, $1 \le i \le n$ be the radicals, the coheight 1 minimal prime ideals in R^*. Then $T(R^*)$ is equal to $T(R^*/I_0) \oplus K_1 \oplus \cdots \oplus K_n$, where K_i is the total quotient ring of R^*/I_i . Then $T(R^*/I_0)$ is integral over R^*/I_0 (cf. [3], (10. 4)), and the integral closure V_i^* of R^*/I_i in K_i is a pseudo valuation ring(cf. [2], (7.6)) and the inverse image of V_i^* under the inclusion $K \to K_i$ $(\to T(R^*/P_i))$ is the Noetherian valuation ring V_i which corresponds to the minimal prime P_i as stated in (5). Thus the integral closure of R^* in $T(R^*)$ is $B' = T(R^*/I_0) \oplus V_1^* \oplus \cdots \oplus V_n^*$, and that of R in $T(R)$ is equal to $B = T(R) \cap V_1 \cap \cdots \cap V_n$. In particular, since a Henselian local domain is unibranched, its completion has no coheight 1 minimal prime divisors if its dimension ≥ 2. Thus the zero ideal of the Henzelization of R has n minimal prime divisors which correspond to P_i , $1 \le i \le n$. Letting r be the number of prime divisors which are of coheight ≥ 2, we have the latter part of our assertion.

Next, for any Noetherian domain R, we consider the ring $R^{(1)}$ which is the intersection of all R_P with height $P = 1$, $P \in SpecR$.

(7) *If* R *is a semi-local domain such that the zero ideal of the completion* R^* *has no embedded prime divisors and only finitely many essential primes have height*>1, *then there is a finite integral extension* A *of*

R in K such that $A^{(1)}$ is finite integral over R.

In fact $R^{(1)}$ is finite integral over R if and only if there are no essential prime divisors of height > 1 and only finitely many embedded prime divisors of nonzero elements in R. In particular if R is semi-local, then R satisfies the latter condition(cf. [4], (7. 7)). Hence any finite integral extension in K also satisfies this condition. Our assumption implies that every essential prime ideal is an asymptotic prime ideal, that is, an element of $\bar{A}^*(R)$ (cf. (2), (c)). Let R' be the integral closure of R in K. Let $P_1, \cdots, P_s$ be the embedded essential prime divisors of nonzero elements in R. Then there are only finitely many prime ideals $P'_1, \cdots, P'_m$ in R' whose contractions to R are the P_i, $1 \leq i \leq s$. It is easy to see that there is a finite integral extension A in K such that for each prime ideal P'_k, P'_k is the only prime ideal which lies over $P'_k \cap A$. Let P be a essential prime ideal of A. Suppose that P is embedded. Then we see that $P \cap R = P_i$ for some i. Let P' be a prime ideal of R' which lies over P. Then $P' = P'_k$ for some P'_k. Thus we have $P = P'_k \cap A$ for some k. From the construction of A, we have height $P =$ height $P'_k = 1$, a contradiction. Thus every essential prime ideal in A is height one, and this proves our assertion.

(8) *Remark. There is a 2-dimensional local domain S whose integral closure is finite integral over S and has two maximal ideals of height* 1 *and* 2 (cf. [5], p. 205). *Thus its completion has a coheight 1 minimal prime ideal which is a reduced component of zero, and the maximal ideal is an essential asymptotic prime ideal. Consequently, the ring $S^{(1)}$ is not finite integral over S. On the other hand, there is a 2-dimensional local domain T whose integral closure is a regular local ring, not finite integral over T* ([1]). *In this example, the maximal ideal of T is essential, and there is no such finite integral extension A as mentioned above.*

Next we discuss some results concerning normalization of homogeneous domains. Let V be a projective variety defined over an algebraically closed field k and let $R = \sum R_m$ its homogeneous coordinate ring, where R_m is the homogeneous part of degree m in R. Let M be the maximal ideal generated by $R_1 = kx_0 + \cdots + kx_n$.

(9) *Let S be the $M-$transform of R, the ideal transform. Then S is contained in R', the integral closure of R, and V is projectively normal if and only if V is normal and $(x_i) : M = (x_i)$ for some $x_i \neq 0$.*

The first part is easy. In fact, R' is finite integral over R, and there is no height 1 prime ideal lying over M. (If such prime ideal N exists, then R'/N should be integral over k.) Since S is the intersection of all localizations R_P for prime ideals $P \neq M$, S is integral over R, hence is contained in R'. The normality of V implies $S = R'$ (cf. [7], p. 175). On the other hand, if V is projectively normal, then R is normal, and M is not a prime divisor of regular elements. Conversely suppose that V is normal and $(x_i) : M = (x_i)$ for some $x_i \neq 0$. Then for any non-zero element $y \in R$, M is not a prime divisor of (y). Thus we have $R = S = R'$ and V is projectively normal.

As is well known, under the the assumption that V is normal, the linear system of hypersurface sections of degree m is complete for sufficiently large m ($R_m = S_m$ for all large m). As for this direction, we note the following known result.

(10). *With the same notation as in (9), suppose that V is nonsingular of codimension one. If U be a projective variety which is defined by a homogeneous domain T with $R \subseteq T \subseteq R'$, then the degree of U is equal to that of V. Also, V is projectively normal, that is, $R = R'$ if and only if any linear system of hypersurface sections is complete, or equivalently for every $d-$ uple embedding W of V, W is not a non-trivial projection from a projective variety of same dimension and same degree.*

Let $T = k[y_0, \cdots, y_m]$ and $N = Ty_1 + \cdots + Ty_m$. As is well known, the extended ideal MT is an $N-$ primary ideal and a reduction of N. Then the localization T_N is finite integral over R_M, and their multiplicities are equal to the degrees of the corresponding varieties. By the extension formula of multiplicities, we have deg $V =$ deg U. The later part is a well-known result.

Finally we describe some results related to the concepts of normalization and complete modules. In the above statement, the homogeneous coordinate ring $R = \sum R_m$ is normal if and only if each $k-$ module R_m is complete(cf. [7], p. 350). Now, let $R \subseteq K$ be rings, let I be an $R-$ module contained in K, let $H = R[It]$ with t an indeterminate and let $F = K[t]$. Then the integral closure $\overline{H}$ of H in F is a graded subring of F, and $zt^r \in \overline{H}$ if and only if $z \in K$ is integral over I^r. Thus $\overline{H} = \sum \overline{I^r}\, t^r$, where $\overline{I^0} = \overline{R}$ ($resp. \overline{I^r}$) is the integral closure of of R ($resp. I^r$) in K.

We assume that K is a field. Let V be a valuation ring of $K(t)$ which contains H. Then $V' = V \cap K$ is a valuation ring of K which contains R and V contains $V'I^r t^t$. Conversely, for any valuation ring V' of K which contains R, if $at^r \notin V'I^r t^t$, then $a \in K$ is not integral over I^r and at^r is not integral over H. Thus there is a valuation ring V of $K(t)$ with $at^r \notin V$ and $H \subseteq V$. The integral closure $\overline{H}$ of H in $K(t)$ is graded, and the completion of I^r is defined to be the intersection of all $V'I^r$ mentioned above. Hence the integral closure of I^r in K is equal to $\cap V'I^r = K \cap t^{-r}V$, where V' ($resp, V$) runs over all valuation rings of K containing R ($resp.$ *in* $K(t)$ *containing* H.)

We close this note with a brief remark on integral closures of ideals . Here we introduce a useful ring $R(I)$ which is derived from a Noetherian domain R and a non-zero ideal I. Let $R(I)$ be the set $\cup I^s a^{-1}$, where a runs over all superficial elements of degree s for I, the first neighbourhood ring with respect to I, a slight generalization of that firstly introduced by Northcott. From the definition we have the following :

(11) *$R(I)$ is a normal domain if and only if $R(I)$ is a finite intersection of discrete valuation rings. Suppose that R is normal. Then $R(I)$ is normal if and only if I^n is integrally closed in R for all large n. Further, if $R(I)$ is local, then I^n is a complete primary ideal for all large n.*

(12) *$R[It]$ is normal if and only if R and $R(I)$ are normal and there is an element a in I such that $I^n : a = I^{n-1}$ for all $n \geq 1$.*

The proof of these statements are easy. First assume that R is normal. $R(I)$ is semi-local and each maximal ideal is a prime divisor of the principal ideal $IR(I)$. Hence, if $R(I)$ is normal, then it is a semi-local principal ideal domain. From the definition of $R(I)$, $I^nR(I) \cap R = I^n$ for all large n and each $I^nR(I)$ is normal in K. Thus I^n is normal in K for all large n. Conversely assume that I^n is normal in R for all large n. Since $R(I) = R(I^r)$, $r \geq 1$, we see that $R(I)$ is normal. Next assume that $R[It]$ is normal. Then R and $R(I)$ are normal. Since $S = R[u, It]$, $u = t^{-1}$ is normal and $S/uS = R[It]/IR[It]$, $IR[It]$ has no embedded prime divisors. Hence there is a strongly surperficial element a as mentioned above. Conversely assume that R and $R(I)$ are normal and there is an element a such that $I^n : a = I^{n-1}$ for all $n \geq 1$. In this case we have $I^nR(I) \cap R = I^n$ for all $n \geq 1$. From the assumption I^n is normal in K for all $n \geq 1$. Since R is normal, we see that $R[It]$ is normal.

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